

CHAPTER (1) : NUMERICAL ANALYSIS

Curve Fitting

Method 1 : Least Mean Square

يكون معطى جدول ، ويمثل معادله خط مستقيم مثلا $y = a + bx$ نقوم باستنتاج المعادلات كالآتي

1- نعمل $\sum$ واللي مفهوش x نضربه في n

2- نضرب في x ثم نعمل $\sum$ زي (1)

$y = ax + b$		$y = a + bx$	
$\sum y = a \sum x + nb$	$\sum xy = a \sum x^2 + b \sum x$	$\sum y = na + b \sum x$	$\sum xy = a \sum x + b \sum x^2$

Root Mean Square Error (RMSE)

$$D_{\text{الخطا لكل نقطة}} = |y_i - y_{i(\text{approx})}|$$

$$RMSE = \sqrt{\frac{\sum D_i^2}{n}}$$

Some Common Ideas

$$(1) y = ax^b \Rightarrow \ln y = \ln a + b \ln x = Y = A + bX$$

$$(2) y = \frac{x}{ax + b} \Rightarrow \frac{1}{y} = a + \frac{b}{x} = Y = a + bX$$

$$(3) y = a + bx^2 \Rightarrow X = x^2$$

$$(4) y = \frac{1}{a + b \cos(\theta)} \Rightarrow \frac{1}{y} = 1 + b \cos(\theta) \Rightarrow \frac{1}{y} = Y, \cos(\theta) = X$$

General Form

تكون المعادله على الصوره

$$y = a_0 \phi_0(x) + a_1 \phi_1(x) + \dots + a_n \phi_n(x)$$

$$y = \sum a \phi(x)$$

ونحلها عن طريق تكوين مصفوفه بالمعاملات تكون مصفوفه مربعه نرص الدوال بالطول وبالعرض ويضربهم في بعض

$$A = \begin{bmatrix} \sum \phi_0^2 & \sum \phi_0 \phi_1 & \sum \phi_0 \phi_2 & \sum \phi_0 \phi_3 \\ \sum \phi_1 \phi_0 & \sum \phi_1^2 & \sum \phi_1 \phi_2 & \sum \phi_1 \phi_3 \\ \sum \phi_2 \phi_0 & \sum \phi_2 \phi_1 & \sum \phi_2^2 & \sum \phi_2 \phi_3 \\ \sum \phi_3 \phi_0 & \sum \phi_3 \phi_1 & \sum \phi_3 \phi_2 & \sum \phi_3^2 \end{bmatrix} * \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \sum y_0 \phi_0 \\ \sum y_1 \phi_1 \\ \sum y_2 \phi_2 \\ \sum y_3 \phi_3 \end{pmatrix}$$

Interpolation :

1- Lagrange Method

عدد نقاط الجدول المعطى نسميه على طول $n+1$ ونكون درجه كثيره الحدود n =>

والقانون هو :

$$P_{\leq n}(x) = y_0 L_0(x) + y_1 L_1(x) + \dots + y_n L_n(x)$$

$$L_0 = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}$$

$$L_1 = \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}$$

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$$L_2 = \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$$

و عشان نجيب قيمة الدالة عند نقطه معينه نشيل كل x ونضع القيمه دي

ملحوظات

- اذا طلب كثيره حدود من درجه اقل من n نحذف خانات من نهايه الجدول ونطبق القانون

- اذا طلب كثيره حدود من درجه اكبر من n نطبق curve fitting ثم نوجد داله و نزود نقاط للجدول ثم نطبق ف لاجرانج

- اذا طلب قيمه جذور ل x نجري لاجرانج لل y وليس لل x وبعد الحصول على كثيره الحدود نضع $y=0$

$$Error \leq \frac{\max |f^{n+1}(x)|}{(n+1)!} | (x - x_0)(x - x_1)(x - x_2) \dots (x - x_n) |$$

$$\text{for specific point } E = |f(x) - P_n(x)|$$

Newton Divided method

$$P_n(x) = y_0 + \delta(x - x_0) + \delta^2(x - x_0)(x - x_1) + \delta^3(x - x_0)(x - x_1)(x - x_2)$$

For example :

X	Y	δ	δ^2	δ^3
1	2	$\frac{8-2}{2-1} = 6$	$\frac{1-6}{4-1} = -\frac{5}{3}$	$0 - \left(-\frac{5}{3}\right) = \frac{1}{3}$
2	8	$\frac{10-8}{10-2} = 1$	$\frac{1-1}{6-2} = 0$	
4	10	$\frac{12-10}{12-4} = 1$		
6	12			

Operators

- 1- Shift Operator $E^n f(x) = f(x + nh)$
- 2- Forward Operator $\Delta f(x) = f(x + h) - f(x)$; $\Delta = E - 1$
- 3- Backward Operator $\nabla f(x) = f(x) - f(x - h)$; $\nabla = 1 - E^{-1}$
- 4- Center Operator $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$; $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$
- 5- Mean Operator $\mu f(x) = \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right] = \frac{1}{2} \delta v$; $\delta = \frac{E^{\frac{1}{2}} - E^{-\frac{1}{2}}}{2}$

$$\Delta(f \cdot g) = \Delta f \cdot g_{r+1} + f \cdot \Delta g$$

$$\Delta\left(\frac{f}{g}\right) = \frac{g\Delta f - \Delta g \cdot f}{g_{r+1}g_r}$$

لايجاد قيمة مفقوده في جدول باستخدام operator

- 1- نضع هذه القيمة ب α
- 2- نضع $\Delta^n y_0 = 0$
- 3- نحول Δ إلى E باستخدام الخواص
- 4- نفك ببساطة ونعوض من الجدول فينتج معادله ف α نحلها
- 5- لو في مجهولين نحل معادلتين واحده في $\Delta^n y_0 = 0$ وواحد $\Delta^{n+1} y_0 = 0$

Derivatives by operators form :

$$D = \frac{d}{dx}$$

$$E = e^{hD}$$

$$D = \frac{2}{h} \sinh^{-1} \left(\frac{\delta}{2} \right)$$

$$D = \frac{1}{h} \left[\Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \dots \right]$$

$$D = \frac{1}{h} \left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \dots \right]$$

عشان اثباتات القانونين اللي فوق Flash back

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\text{by integegrating } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\ln(1-x) = - \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right]$$

ملحوظه : قيمه المشتقه الأولى باستخدام ال Δ تكون ادق عند النقاط الأقرب لبدايه الجدول ، وباستخدام ال ∇ تكون ادق عند القيم الأقرب لنهايه الجدول .

Newton Forward

$$P = \frac{x - x_0}{h}$$

$$p_{n(x)} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

Newton Backward

$$P = \frac{x - x_n}{h}$$

$$p_{n(x)} = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots$$

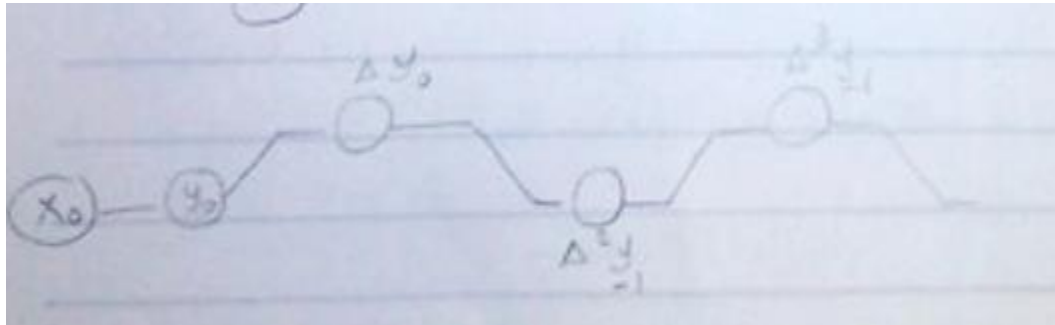
Gauss Forward

$$P = \frac{x - x_0}{h}$$

زوجي يمين ، فردي شمال 0,0,-1,-1,-2,-2

$$P_n(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_{-1} + \frac{(P+1)P(P-1)}{3!}\Delta^3 y_{-1}$$

Gauss Backward

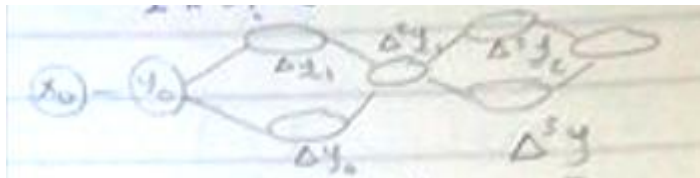


$$P = \frac{x - x_0}{h}$$

زوجي شمال ، فردي يمين 0,-1,-1,-2,-2

$$P_n(x) = y_0 + P\Delta y_{-1} + \frac{(P+1)P}{2!}\Delta^2 y_{-1} + \frac{(P+1)P(P-1)}{3!}\Delta^3 y_{-2} + \dots$$

Stirling Form



$$P = \frac{x - x_0}{h}$$

$$\frac{1}{2} [G.F. + G.B.]$$

$$P_n(x) = y_0 + \frac{P}{2} [\Delta y_0 + \Delta y_{-1}] + \frac{1}{2}\Delta^2 y_{-1} \left[\frac{P(P-1)}{2!} + \frac{P(P+1)}{2!} \right] + \frac{(P+1)P(P-1)}{2 \cdot 3!} [\Delta^3 y_{-1} + \Delta^3 y_{-2}]$$

Linear Spline (1 st degree)	Quadratic Spline (2 nd degree)	Cubic Spline (3 rd degree)
$S_i(x) = y_i + \frac{\overbrace{y_{i+1} - y_i}^{\text{Slope}}}{x_{i+1} - x_i} (x - x_i)$	$S_i(x) = a_i x^2 + b_i x + c_i$ $S'_i(x) = 2a_i x + b_i$ لايجاد a_i, b_i, c_i يجب ان يتحقق شرط الاتصال للداله ومشتقاتها 1- $a_0 = 0$ 2- نعوض بقيم الجدول في الداله 3- نساوي قيم ال S' عند النقط الفاصله	$S_i(x) = a_i(x - x_i)^3 + b_i(x - x_i) + c_i(x - x_i) + d_i$ $i = 1, 2, 3, \dots, n - 1$ $d_i = y_i$ $a_i = \frac{M_{i+1} - M_i}{6h}$ $b_i = \frac{M_i}{2}$ $c_i = \frac{y_{i+1} - y_i}{h} - \frac{h}{6}(M_{i+1} + 2M_i)$ M_i معادله الحصول على $M_i + 4M_{i+1} + M_{i+2} = \frac{6}{h^2} [y_i - 2y_{i+1} + y_{i+2}]$ $i = 0, 1, 2, \dots, n - 1$ $M_1 = M_n = 0$

Approximate for first derivatives

a) Divided Difference Methods

Backward divided difference	Central divided difference	Forward divided difference
$f'(x) = \frac{f(x) - f(x-h)}{h}$	$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$	$f'(x) = \frac{f(x+h) - f(x)}{h}$
$T.E \leq \left \frac{hf''(\rho)}{2} \right $	$T.E \leq \left \frac{hf'''(\rho)}{3!} \right $	$T.E \leq \left \frac{hf''(\rho)}{2} \right $

b) Richardson Extrapolation

$$a(\text{حاجه}) = \frac{f(x + \text{حاجه}) - f(x - \text{حاجه})}{2\text{حاجه}}$$

$$a(h) = \frac{f(x+h) - f(x-h)}{2h}$$

كلما قللنا ال h إلى $h/2$ او $h/4$ او $h/6$ كلما زادت الدقه

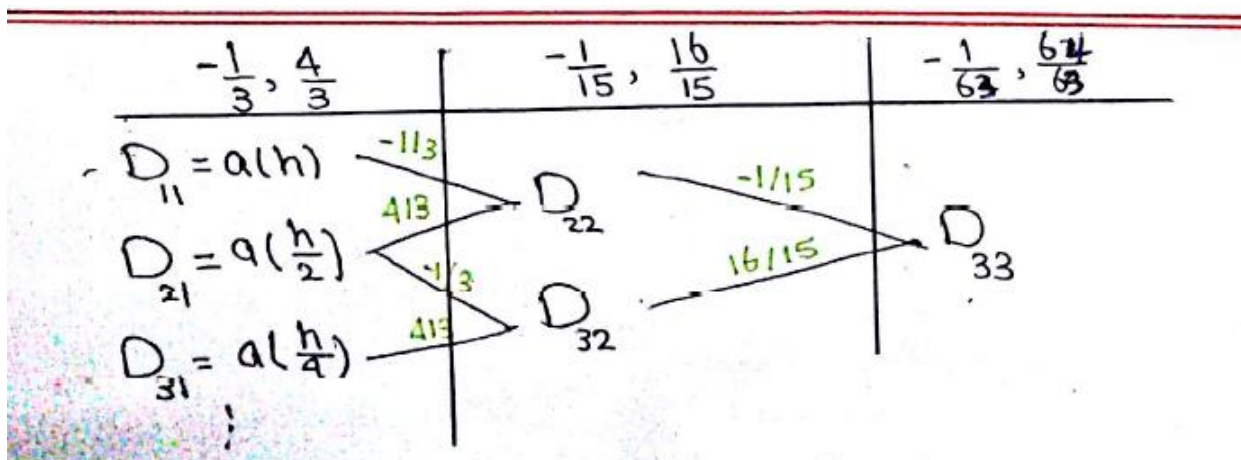
طريقه الحل

1- احسب قيم D_{11} و D_{21} و D_{31} كالآتي

$$D_{n1} = a\left(\frac{h}{2^{n-1}}\right); D_{11} = a(h); D_{21} = a\left(\frac{h}{2}\right); D_{31} = a\left(\frac{h}{4}\right)$$

2- احسب قيم D_{12} و D_{22} و D_{33} من القانون الاتي او من الجدول الاتي برضه D :

$$D_{n,m+1} = \frac{4^m}{4^m - 1} D_{n,m} - \frac{1}{4^m - 1} D_{n-1,m}$$



$$T.E \leq O(h^{2m+2})$$

Approximate for second derivatives

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$T.E \leq \left| \frac{h^2}{12} f'''(\rho) \right|$$

Numerical Integration

$$I = \int_a^b f(x) dx \quad \text{لحل}$$

نقسم فتره التكامل لعدد n من الفترات وتكون الخطوه $h = \frac{b-a}{n}$

ثم نحسب قيمه $f(x)$ عند كل نقطه من نقاط الفتره المقسمه بالطرق الاتيه

Trapezoidal	Simpson
$I = \frac{h}{2} [y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]$ $T.E \leq \frac{(b-a)h^2}{12} M_2 \quad \text{or} \quad \frac{(b-a)^3}{12n^2} M_2$	$I = \frac{h}{3} [y_0 + y_n + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)]$ $T.E \leq \frac{(b-a)h^4}{180} M_4 \quad \text{or} \quad \frac{(b-a)^5}{180n^4} M_4$
Gaussian Quadrature	Weddle
الأول نحول التكامل $\int_a^b f(x)dx$ الى $\int_{-1}^1 f(z)dz$ من الاتي : $\frac{x-a}{b-a} = \frac{z+1}{2}$ $x = a + \frac{(z+1)(b-a)}{2}$ $dx = \frac{b-a}{2} dz$ ثم نعوض ف القانون $\int_{-1}^1 f(x)dx = \sum W_i f(x_i)$ حيث لو two point $W_1 = 1 ; x_1 = \frac{-1}{\sqrt{3}}$ $W_2 = 1 ; x_2 = \frac{1}{\sqrt{3}}$ Three point	الكود 151 6 151 For n= 6 $I_N = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$ For n = 12 151 6 151 151 6 151 $I_N = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$

$W_1 = \frac{5}{9} ; x_1 = -\sqrt{\frac{3}{5}}$ $W_2 = \frac{8}{9} ; x_2 = 0$ $W_3 = \frac{5}{9} ; x_3 = \sqrt{\frac{3}{5}}$	
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Romberg

$$h = \frac{b-a}{2^{k-1}}$$

$$R_{k,1} = \frac{b-a}{2^k} [f(a) + f(b)] + \frac{(b-a)}{2^{k-1}} \sum_{i=1}^{2^{k-1}-1} f\left(a + \frac{b-a}{2^{k-1}} i\right)$$

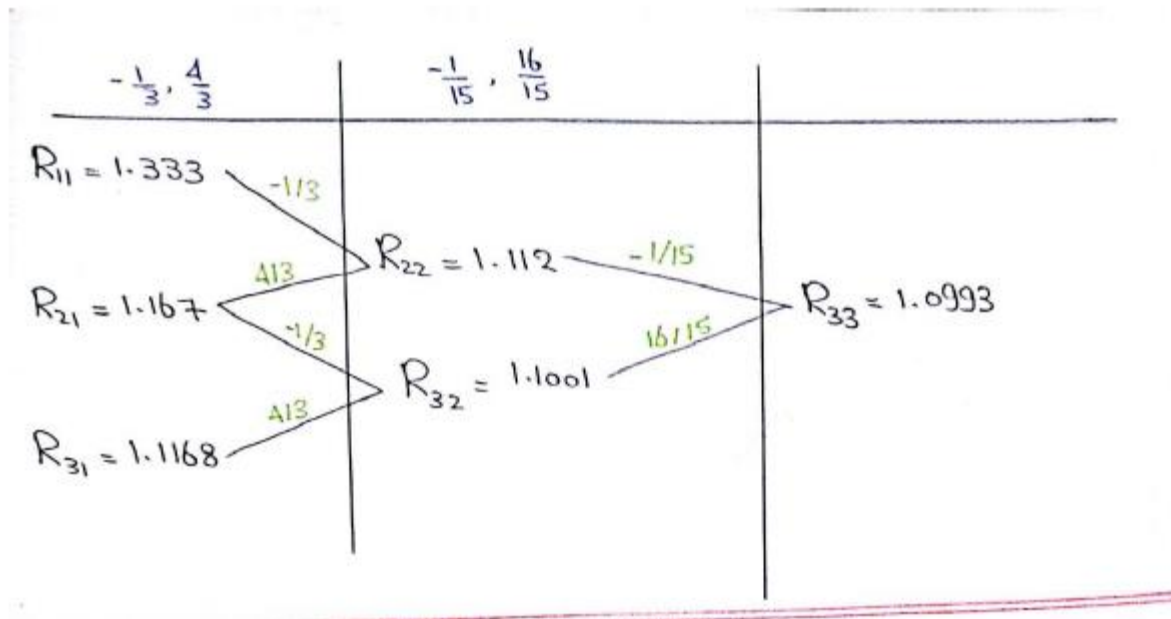
Therefore

$$R_{11} = \frac{b-a}{2} [f(a) - f(b)]$$

$$R_{21} = \frac{R_{11}}{2} + \frac{b-a}{2} f\left(a + \frac{b-a}{2}\right)$$

$$R_{31} = \frac{R_{21}}{2} + \frac{b-a}{4} \left(f\left(a + \frac{b-a}{4}\right) + f\left(a + \frac{3(b-a)}{4}\right) \right)$$

ثم نستخدم الجدول الاتي لاستخراج باقي قيم التكامل الادق



Numerical Solution for ODE

تكون المعطيات هي

$$y' = f(x, y) \quad ; \quad y(x_0) = y_0 \quad ; \quad h = \frac{x_n - x_0}{n}$$

(one step)

1- Euler Method

$$y_{i+1} = y_i + h f_i$$

Global error

$$|E_n| = \frac{hk}{2M} [(1 + hM)^n - 1]$$

2- Taylor Method

2nd Order

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2!} f'_i$$

3rd Order

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2!} f'_i + \frac{h^3}{3!} f''_i$$

3- 4th Order RK (Runge Kutta)

$$y_{i+1} = y_i + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = f(x_i, y_i)$$

$$K_2 = f\left(x + \frac{h}{2}, y + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(x + \frac{h}{2}, y + \frac{h}{2}K_2\right)$$

$$K_4 = f(x + h, y + hK_3)$$

4- Modified Euler (Corrector – Predictor)

Predictor Step

$$y_{i+1}^P = y_i + hf_i = y_{i+1}^{c(0)}$$

$$y_{i+1}^{c(r+1)} = y_i + \frac{h}{2} \left(f_i(x_i, y_i) + f(x_{i+1}, y_{i+1}^{c(r)}) \right)$$

When you have a 2nd Order ODE we convert it into a 1st order system of equations then we solve it

• Adams - Bashforth Explicit method

$$y_{n+1} = y_n + h \left[P_n + \frac{1}{2} \nabla P_n + \frac{5}{12} \nabla^2 P_n + \frac{3}{8} \nabla^3 P_n + \dots \right]$$

• القيم "y" قبل "y" المطلوبة يتم حسابها بـ "Euler"

t_i	P_i	∇P	$\nabla^2 P$
0	1.5		
0.3	$\frac{93}{50}$	$\frac{9}{25}$	
0.6	$\frac{537}{250}$	$\frac{36}{125}$	$\frac{-9}{125}$
0.9			

$$\therefore y_3 = y_2 + h \left[P_2 + \frac{1}{2} \nabla P_2 + \frac{5}{12} \nabla^2 P_2 \right]$$

$$\therefore y_3 = 2.1866$$

• linear 2nd order p.d.e

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

• Classification of P.D.E

$$B^2 - 4AC \begin{cases} = 0 & \text{Parabolic} \rightarrow \text{heat Eqn.} \\ < 0 & \text{Elliptic} \rightarrow \text{laplace eqn. (poisson)} \\ > 0 & \text{hyperbolic} \rightarrow \text{wave eqn.} \end{cases}$$

هنا يكون الحل على شكل شبكة "grid" حيث يوجد تغير في الـ "x" و "t".

• Parabolic Eqn. "1-D heat Eqn."

$$u_t(x,t) = \alpha^2 u_{xx}(x,t)$$

$$u_t = \alpha^2 u_{xx}$$

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} \quad \text{F.w}$$

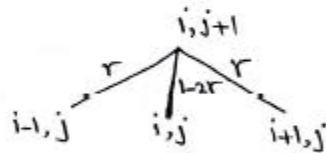
$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h} \quad \text{B.w}$$

$$u_{xx} = \frac{u_{i+1,j} - u_{i,j} + u_{i,j} - u_{i-1,j}}{2h} \quad \text{central}$$

$$0 \leq x \leq L, t \geq 0, u(0,t) = u(L,t) = 0, u(x,0) = P(x) \quad \text{I.C. B.C}$$

• For Conditionally stable $0 \leq r \leq \frac{1}{2} \Rightarrow r = \frac{K\alpha^2}{h^2}$

$$u_{i,j+1} = r u_{i-1,j} + (1-2r) u_{i,j} + r u_{i+1,j}$$



• في هذا النوع من المعادلات تكون الشبكة مفتوحة هذا فوق أو اسفل أو يمين أو يسار.

خطوات الحل

- 1- تقسم الشبكة في الـ x باستخدام الـ "h" وفي الـ t باستخدام الـ "K".
- 2- نضع قيم الحل عند الحدود والـ initial.
- 3- نوجد القيم المطلوبة عند التقاطع.

Proofs

Error by Lagrange

Deduce The error by Lagrange Polynomial

$$|E| \leq \frac{M_{n+1}}{(n+1)!} |(x-x_0)(x-x_1)(x-x_n)|.$$

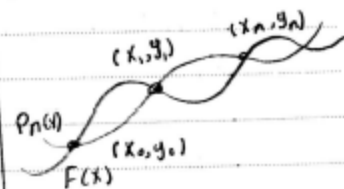
$$u(x) = f(x) - P_n(x) - K \pi_{n+1}(x). \quad (1)$$

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)(x-x_2) \dots (x-x_n).$$

المعادلة رقم (1) عند قيم الجذور وهي

$x_0, x_1, \dots, x_n$ قيمتها = مفر أي أن

لها $n+1$ من الجذور $\Rightarrow u(x)$



إذا أردنا أن نجعل المعادلة لها $n+2$ من الجذور فنسب قيمة K

$$f(x^*) - P_n(x^*) - K \pi_{n+1}(x^*) = 0$$

$$K = \frac{f(x^*) - P_n(x^*)}{\pi_{n+1}(x^*)}$$

$x^n + o$ Ther error terms

$$\frac{d^n}{dx^n} P_n(x) = C n!$$

نضاعل المعادلة (1) من المرات n

$$u(x) = R(x) - C_1(n!) - k[(n+1)!x + C_2] \quad (2)$$

$$\frac{d^n u}{dx^n} = (n+1)!x + C_2$$

$$F(x) - P_n(x) = 0 \quad \leftarrow \text{جميع قيم } x \text{ في الجول}$$

$$F(x) = P_n(x)$$

$$F'(x) = P_n'(x)$$

$$F^{(n)}(x) = P_n^{(n)}(x)$$

$$u^{(n)}(x_i) = u^{(n)}(x_j)$$

نضاعل المعادلة (2) يمكن استنتاج تعطين x_i و x_j المشتقة النونية عندهم متساوية

لوصفنا نقطتين قيمة الدالة متساوية عندهم فانه يوجد نقطة في المنتصف فانه قيمة المشتقة ~~التي~~ عندها = صفر ولذلك نستنتج ان ~~النقطة~~



$$u(x) = 0 \quad x_i < \xi < x_j$$

$$P(x) - k(n+1)! = 0$$

$$k = \frac{P(x)}{(n+1)!}$$

$$u(x) = 0$$

اي ان ξ اذا كان

$$M_{n+1} = \max_{x_i \leq \xi \leq x_j} |P(x)|$$

$$|E| \leq \frac{M_{n+1}}{(n+1)!} |(x-x_0)(x-x_1)\dots(x-x_n)|$$

"Proofs"

Method of Least Squares:-
[1] $y = ax + b$

error $D_i = e_i = y_i - (ax_i + b)$

$$D_i^2 = (y_i - ax_i - b)^2$$

$$S_i = \sum D_i^2 = \sum [y_i - ax_i - b]^2 \Rightarrow 2 \text{ var. } a, b$$

For S_i to be minimum. $\frac{\partial S_i}{\partial a} = 0$, $\frac{\partial S_i}{\partial b} = 0$

$$\frac{\partial S_i}{\partial a} = -2 \sum (y_i - ax_i - b) x_i = 0$$

$$\sum x_i y_i = a \sum x_i^2 + b \sum x_i \rightarrow \textcircled{1}$$

$$\frac{\partial S_i}{\partial b} = -2 \sum (y_i - ax_i - b) = 0$$

$$\sum y_i = a \sum x_i + \sum_{i=1}^n b$$

$$\sum y_i = a \sum x_i + nb \rightarrow \textcircled{2}$$

Solving $\textcircled{1}$ & $\textcircled{2}$ to get a, b where n is no. of points

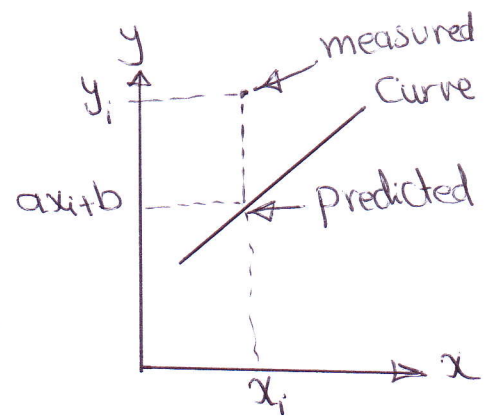
hint

the other form is

$$\sum y_i = na + b \sum x_i$$

$$\sum x_i y_i = a \sum x_i + b \sum x_i^2$$

when fitting in form $y = a + bx$



[2]

General Case:-

$$y = \sum_{i=0}^n a_i \Phi_i(x) = a_0 \Phi_0(x) + a_1 \Phi_1(x) + \dots + a_n \Phi_n(x)$$

$$D_i^2 = [y_i - a_0 \Phi_0(x) - a_1 \Phi_1(x) - \dots - a_n \Phi_n(x)]^2$$

For minimum error $\frac{\partial S_i}{\partial a_i} = 0$, $S_i = \sum D_i^2$, var. $a_0 \dots a_n$

$$\frac{\partial S_i}{\partial a_0} = -2 \sum [y_i - a_0 \Phi_0(x) - \dots - a_n \Phi_n(x)] \Phi_0(x) = 0$$

$$\therefore \sum y_i \Phi_0(x) = a_0 \sum \Phi_0^2(x) + a_1 \sum \Phi_0 \Phi_1(x) + \dots + a_n \sum \Phi_0 \Phi_n(x)$$

$$\frac{\partial S_i}{\partial a_1} = -2 \sum [y_i - a_0 \Phi_0(x) - a_1 \Phi_1(x) - \dots - a_n \Phi_n(x)] \Phi_1(x) = 0$$

$$\therefore \sum y_i \Phi_1(x) = a_0 \sum \Phi_0 \Phi_1(x) + a_1 \sum \Phi_1^2(x) + \dots + a_n \sum \Phi_1 \Phi_n(x)$$

$$\frac{\partial S_i}{\partial a_n} = -2 \sum [y_i - a_0 \Phi_0(x) - \dots - a_n \Phi_n(x)] \Phi_n(x) = 0$$

$$\therefore \sum y_i \Phi_n(x) = a_0 \sum \Phi_0 \Phi_n(x) + \dots + a_n \sum \Phi_n^2(x)$$

Put Eqns in Matrix Form

$$\begin{bmatrix} \sum \Phi_0^2(x) & \sum \Phi_0 \Phi_1(x) & \dots & \sum \Phi_0 \Phi_n(x) \\ \sum \Phi_0 \Phi_1(x) & \sum \Phi_1^2(x) & \dots & \sum \Phi_1 \Phi_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \sum \Phi_0 \Phi_n(x) & \dots & \dots & \sum \Phi_n^2(x) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y_i \Phi_0(x) \\ \sum y_i \Phi_1(x) \\ \vdots \\ \sum y_i \Phi_n(x) \end{bmatrix}$$

Lagrange's interpolation Formula

Let $P_n(x)$ is a polynomial in x of degree n Then.

$$P_n(x) = C_0(x-x_1)(x-x_2)\dots(x-x_n) + C_1(x-x_0)(x-x_2)\dots(x-x_n) \\ + \dots + C_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

where $C_0, C_1, \dots, C_n$ are constants

$$\therefore P_n(x_0) = y_0 \therefore y_0 = C_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

$$C_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

$$\therefore P_n(x_1) = y_1$$

$$\therefore C_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

$$C_n = \frac{y_n}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

$$\therefore P_n(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 \\ + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

Lagrange's Error

Error $E(\bar{x}) = F(\bar{x}) - P_n(\bar{x})$ error at an intermediate $\bar{x}$

$P_n(x_i) = F(x_i)$, $E(x_i) = 0$, $i = 0, 1, \dots, n$ that is there are no errors of interpolation at tabulated points. لا يوجد خطأ في قيم الجدول

Interpolation - Error theorem

- (i) $F(x)$ be $n+1$ times continuously differentiable on closed interval $[a, b]$
- (ii) $x_0, x_1, \dots, x_n$ be $n+1$ distinct points in $[a, b]$
- (iii) $P_n(x)$ be the interpolating polynomial of degree at most n that interpolates $F(x)$ in $[a, b]$

Then For every $\bar{x}$ in $[a, b]$ there exists a number $\xi = \xi(\bar{x})$

in (a, b) such that $E_n(\bar{x}) = F(\bar{x}) - P_n(\bar{x}) = \frac{F^{(n+1)}(\xi(\bar{x}))}{(n+1)!} \prod_{i=0}^n (x - x_i)$

Generalized Rolle's theorem

Suppose $F(x)$ satisfies the following hypotheses:-

- (i) it is continuous on $[a, b]$
- (ii) it is n times differentiable on (a, b)
- (iii) it has $n+1$ distinct zeros in $[a, b]$

Then there exists a number c in (a, b) such that

$$F^{(n)}(c) = 0$$

"Proof"

Define $g(t) = F(t) - P_n(t) - [F(\bar{x}) - P_n(\bar{x})] \frac{(t-x_0)(t-x_1)\dots(t-x_k)(t-x_n)}{(\bar{x}-x_0)(\bar{x}-x_1)\dots(\bar{x}-x_n)}$

Show that Rolle's theorem applied to $g(t)$

① $g(t)$ is $(n+1)$ times Continuously differentiable and so $F(x)$

$\therefore$ (i), (ii) satisfied

② $g(t)$ at $(n+2)$ points $x_0, \dots, x_n, \bar{x}$

at $t = x_k \Rightarrow$ sub into $g(t)$ form

$$g(x_k) = F(x_k) - P_n(x_k) - 0$$

$$= 0 \quad (\text{no error at table points})$$

$\therefore g(x_k) = 0$ at $n+1$ points $x_0, \dots, x_n \rightarrow$ ①
at $t = \bar{x}$

$$g(\bar{x}) = F(\bar{x}) - P_n(\bar{x}) - [F(\bar{x}) - P_n(\bar{x})] \frac{(\bar{x}-x_0)\dots(\bar{x}-x_n)}{(\bar{x}-x_0)\dots(\bar{x}-x_n)}$$

$$g(\bar{x}) = F(\bar{x}) - P_n(\bar{x}) - F(\bar{x}) + P_n(\bar{x}) = 0 \quad \triangle 1$$

$\therefore g(\bar{x}) = 0$ at $n+2$ points $x_0, \dots, x_n \& \bar{x} \rightarrow$ ②

From ① & ② $\therefore$ (iii) Satisfied then Apply Rolle's on

$g(t) \Rightarrow$ there exist $\xi(\bar{x})$ in (a, b) such that $g^{(n+1)}(\xi) = 0$

$$g^{(n+1)}(t) = F^{(n+1)}(t) - P_n^{(n+1)}(t) - [F(\bar{x}) - P_n(\bar{x})] \times$$

$$\frac{d^{n+1}}{dt^{n+1}} \left[\frac{(t-x_0)\dots(t-x_n)}{(\bar{x}-x_0)\dots(\bar{x}-x_n)} \right] \Rightarrow \boxed{3}$$

$\Downarrow$
D

$$\frac{d^{n+1}}{dt^{n+1}} D = \frac{1}{(\bar{x}-x_0) \cdots (\bar{x}-x_n)} \frac{d^{n+1}}{dt^{n+1}} (t-x_0) \cdots (t-x_n)$$

Polynomial in t of $n+1$ degree $\swarrow$

$$= \frac{(n+1)!}{(\bar{x}-x_0) \cdots (\bar{x}-x_n)}$$

$P^{(n+1)}(t) = 0$ since $P_n(x)$ is a polynomial of degree n

Sub into (3)

$$g^{(n+1)}(\bar{x}) = f^{(n+1)}(\bar{x}) - \frac{(n+1)!}{(\bar{x}-x_0) \cdots (\bar{x}-x_n)} [f(\bar{x}) - P_n(\bar{x})]$$

$$t = \xi$$

$$g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{(n+1)!}{(\bar{x}-x_0) \cdots (\bar{x}-x_n)} [f(\bar{x}) - P_n(\bar{x})]$$

$\angle 0$

$$\therefore f(\bar{x}) - P_n(\bar{x}) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (\bar{x}-x_0) \cdots (\bar{x}-x_n)$$

#

Newton's Forward Formula:-

Evaluate $F(x)$ at $x_0 + ph$ where p is a real number.

$$E^p(F(x)) = F(x + ph)$$

$$\therefore F(x_0 + ph) = E^p(F(x_0)) = (1 + \Delta)^p F(x_0)$$

$$\& F(x_0) = y_0, \quad x = x_0 + ph \quad \downarrow \text{Binomial theorem}$$

$$\therefore F(x_0 + ph) =$$

$$(1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots) y_0$$

$$\approx y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

Newton's Backward Formula:-

Extrapolate value of F^n a short distance from y_n

$$F(x_n + ph) = E^p F(x_n) = (E^{-1})^{-p} F(x_n) = (1 - \nabla)^{-p} F(x_n)$$

$$\therefore F(x_n + ph) =$$

$$\left[1 + p\nabla + \frac{-p(-p-1)}{2!} \nabla^2 + \frac{-p(-p-1)(-p-2)}{3!} (-\nabla)^3 + \dots \right] y_n$$

$$\approx y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots$$

$$x = x_n + ph, \quad F(x_n) = y_n$$

Richardson Extrapolation Form:-

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2!} F''(x) + \frac{h^3}{3!} F'''(x) + \frac{h^4}{4!} F^{(4)}(x) + \frac{h^5}{5!} F^{(5)}(x) + \dots$$

$$F(x-h) = F(x) - hF'(x) + \frac{h^2}{2!} F''(x) - \frac{h^3}{3!} F'''(x) + \frac{h^4}{4!} F^{(4)}(x) - \frac{h^5}{5!} F^{(5)}(x) + \dots$$

$$F(x+h) - F(x-h) = 2hF'(x) + 2\frac{h^3}{3!} F'''(x) + 2\frac{h^5}{5!} F^{(5)}(x) + \dots$$

$$F'(x) = \frac{F(x+h) - F(x-h)}{2h} - \frac{h^2}{3!} F'''(x) - \frac{h^4}{5!} F^{(5)}(x) - \dots$$

$$\text{let } \frac{F(x+h) - F(x-h)}{2h} = a(h), \quad C_1 = -\frac{F'''(x)}{3!}, \quad C_2 = -\frac{F^{(5)}(x)}{5!}$$

$$\therefore F'(x) = a(h) + C_1 h^2 + C_2 h^4 + C_3 h^6 \rightarrow \textcircled{1}$$

Replace h by $h/2$ to reduce truncation error

$$F'(x) = a(h/2) + C_1 (h/2)^2 + C_2 (h/2)^4 + \dots \rightarrow \textcircled{2}$$

$$4 \times \textcircled{2} - \textcircled{1} \Rightarrow \text{T.E} \propto O(h^4) \text{ instead of } O(h^2)$$

$$4F'(x) - F'(x) = 4a(h/2) - a(h) + \frac{1}{4}C_2 h^4 - C_2 h^4$$

$$F'(x) = \frac{4}{3}a(h/2) - \frac{1}{3}a(h) - \frac{1}{4}C_2 h^4 \rightarrow \textcircled{3}$$

Repeat this operation and solve $h/2$ with $h/4$ results

③ $\frac{h}{2} \rightarrow h$ ابدل $\frac{h}{2}$ بـ h

$$F'(x) = \frac{4}{3}a(h/4) - \frac{1}{3}a(h/2) - \frac{1}{4}C_2 (h/2)^4 \rightarrow \textcircled{4}$$

From the eq^{ns} yields $\Rightarrow D_{11} = a(h), D_{21} = a(h/2), D_{31} = a(h/4)$

$$D_{n1} = a\left(\frac{h}{2^{n-1}}\right)$$

From (3)

$$f'(x) = \frac{4}{3} D_{21} - \frac{1}{3} D_{11} = D_{2,2}$$

From (4)

$$f'(x) = \frac{4}{3} D_{31} - \frac{1}{3} D_{21} = D_{3,2}$$

$$\therefore D_{n,m+1} = \frac{4^m}{4^m - 1} D_{n,m} - \frac{1}{4^m - 1} D_{n-1,m}$$

now Eqⁿ (3),(4) can be rewritten as

$$f'(x) = D_{22} - \frac{1}{4} c_2 h^4 \rightarrow (5)$$

$$f'(x) = D_{3,2} - \frac{1}{64} c_2 h^4 \rightarrow (6)$$

الفكرة محاولة التخلص من h^4
ليكون $T.E \propto o(h^6)$ لزيادة
الدقة

$$16 \times (6) - (5)$$

$$15 f'(x) = 16 D_{3,2} - D_{22} + o(h^6)$$

$$f'(x) = \frac{16}{15} D_{3,2} - \frac{1}{15} D_{22} + o(h^6)$$

$$D_{33} = \frac{16}{15} D_{3,2} - \frac{1}{15} D_{22}$$

$\therefore$ the relations used is true $\forall m, n$ #

(C) Newton's Divided Difference Formula

Ex. 11

$$F[x_{i+1}, x_i] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \quad \& \quad F[x_{i+2}, x_{i+1}, x_i] = \frac{F[x_{i+2}, x_{i+1}] - F[x_{i+1}, x_i]}{x_{i+2} - x_i}$$

"Procr"

$$\text{let } P_n(x) = C_0 + C_1(x-x_0) + C_2(x-x_0)(x-x_1) + \dots$$

$$\text{at } x = x_0, \quad y = y_0$$

$$\therefore P_n(x_0) = C_0 = y_0$$

$$\text{at } x = x_1, \quad y = y_1$$

$$P_n(x_1) = y_0 + C_1(x_1 - x_0) + 0 = y_1$$

$$C_1 = \frac{y_1 - y_0}{x_1 - x_0} = F[x_1, x_0]$$

$$P_n(x) = y_0 + F[x_1, x_0](x - x_0) + C_2(x - x_0)(x - x_1) + \dots$$

$$\text{at } x = x_2, \quad y = y_2$$

$$y_2 = y_0 + (x_2 - x_0) F[x_1, x_0] + C_2(x_2 - x_0)(x_2 - x_1)$$

$$C_2 = \frac{y_2 - y_0 - (x_2 - x_0) \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)(x_2 - x_1)}$$

$$C_2 = \frac{y_2}{(x_2-x_0)(x_2-x_1)} - \frac{y_0}{(x_2-x_0)(x_2-x_1)} - \frac{y_1}{(x_2-x_1)(x_1-x_0)} \\ + \frac{y_0}{(x_2-x_1)(x_1-x_0)}$$

$$C_2 = \frac{y_2}{(x_2-x_0)(x_2-x_1)} - \frac{y_1}{(x_2-x_1)(x_1-x_0)} + \frac{y_0}{\cancel{x_2-x_1}} \left[\frac{-\cancel{x_2-x_0} - \cancel{x_1} + \cancel{x_0}}{(x_2-x_0)(x_1-x_0)} \right] \\ = \frac{y_2}{(x_2-x_0)(x_2-x_1)} - \frac{y_1}{(x_2-x_1)(x_1-x_0)} + \frac{y_0}{(x_2-x_0)(x_1-x_0)}$$

$$F[x_2, x_1, x_0] = \frac{F[x_2, x_1] - F[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0} \left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0} \right]$$

$$= \frac{y_2}{(x_2-x_0)(x_2-x_1)} - \frac{y_1}{x_2-x_0} \left[\frac{\cancel{x_1-x_0} + \cancel{x_2-x_1}}{(x_2-x_1)(x_1-x_0)} \right] + \frac{y_0}{(x_2-x_0)(x_1-x_0)}$$

$$= \frac{y_2}{(x_2-x_0)(x_2-x_1)} - \frac{y_1}{x_2-x_0} + \frac{y_0}{(x_2-x_0)(x_1-x_0)}$$

$$= C_2$$

$$\therefore P_n(x) = y_0 + (x-x_0) F[x_0, x_1] + (x-x_0)(x-x_1)$$

$$F[x_0, x_1, x_2] + \dots \#$$

Trapezoidal Rule:-

$$I = \int_a^b f(x) dx$$

$I \rightarrow$ Area under the Curve

$$I = \sum I_i = \sum A_i$$

$$h = \frac{b-a}{n}, n: \text{no. of subintervals}$$

$$\text{let } f(x) = F'(x)$$

$$\therefore I_i = \int_{x_{i-1}}^{x_i} f(x) dx = F(x_i) - F(x_{i-1}) = F_i - F_{i-1}$$

From Taylor's:-

$$F_{i-1} = F_i - h F_i' + \frac{h^2}{2!} F_i'' - \frac{h^3}{3!} F_i''' + \dots$$

$$\therefore I_i = F_i - \left[F_i - h F_i' + \frac{h^2}{2!} F_i'' - \frac{h^3}{3!} F_i''' + \dots \right]$$

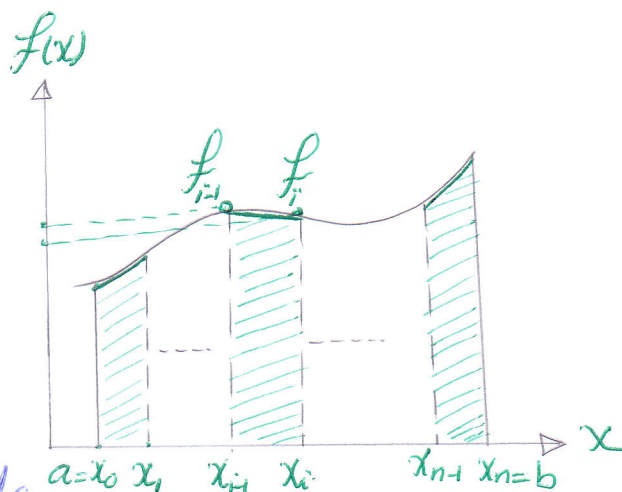
$$I_i = h F_i' - \frac{h^2}{2!} F_i'' + \frac{h^3}{3!} F_i''' + \dots \Rightarrow f_i = F_i'$$

$$= h f_i - \frac{h^2}{2!} f_i' + \frac{h^3}{3!} f_i'' + \dots \rightarrow \textcircled{1}$$

$$\therefore f_{i-1} = f_i - h f_i' + \frac{h^2}{2!} f_i'' - \dots$$

$$f_i' = \frac{1}{h} [f_i - f_{i-1}] + \frac{h}{2} f_i'' \quad \text{sub into } \textcircled{1}$$

$$I_i = h f_i - \frac{h^2}{2!} \left[\frac{f_i - f_{i-1}}{h} + \frac{h}{2} f_i'' \right] + \frac{h^3}{3!} f_i'' + \dots$$



$$I_i = \frac{h}{2} [f_i + f_{i-1}] - \underbrace{\frac{h^3}{4} f_i'' + \frac{h^3}{6} f_i''}_{-\frac{2}{24} h^3 f_i''}$$

$$I_i = \frac{h}{2} [f_i + f_{i-1}] - \frac{h^3}{12} f_i''(s) \quad \hookrightarrow \text{T.E at a one interval}$$

$$I = \sum_i^n I_i = I_1 + I_2 + \dots + I_n \quad \rightarrow n \text{ interval}$$

$$= \frac{h}{2} [f_1 + f_0] + \frac{h}{2} [f_1 + f_2] + \dots + \frac{h}{2} [f_{n-1} + f_n] \\ - \frac{n h^3}{12} f_i''(s) \quad \hookrightarrow \text{T.E at } n \text{ intervals}$$

$$I = \frac{h}{2} [f_0 + f_n + 2(f_1 + \dots + f_{n-1})] - \frac{n h^3}{12} f_i''(s)$$

$$\text{let } f_i''(s) = \max_{s \in [x_0, x_n]} |f''(s)| = M_2, \quad h = \frac{b-a}{n}$$

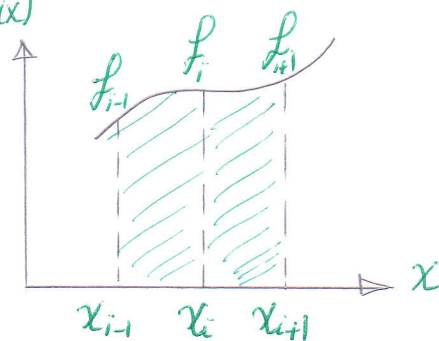
$$\therefore \text{T.E} \leq \frac{(b-a)^3}{12n^2} M_2 \quad \#$$

Simpson's Rule:-

divide each interval in Trapezoidal to 2-intervals $f(x)$

$n=2m$ (even no. of intervals)

$$I = \int_a^b f(x) dx$$



$$I_i = \int_{x_{i-1}}^{x_{i+1}} f(x) dx = F(x_{i+1}) - F(x_{i-1}) = F_{i+1} - F_{i-1}$$

Taylor

$$F_{i+1} = F_i + h F_i' + \frac{h^2}{2} F_i'' + \frac{h^3}{3!} F_i''' + \frac{h^4}{4!} F_i^{(4)} + \frac{h^5}{5!} F_i^{(5)}$$

$$F_{i-1} = F_i - h F_i' + \frac{h^2}{2} F_i'' - \frac{h^3}{3!} F_i''' + \frac{h^4}{4!} F_i^{(4)} - \frac{h^5}{5!} F_i^{(5)}$$

$$I_i = 2h F_i' + \frac{h^3}{3} F_i''' + \frac{h^5}{60} F_i^{(5)}$$

$$f_i = F_i' \rightarrow I_i = 2h f_i + \frac{h^3}{3} f_i''' + \frac{h^5}{60} f_i^{(5)} \rightarrow \textcircled{1}$$

Remember f_i''

$$f_{i+1} = f_i + h f_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{3!} f_i''' + \frac{h^4}{4!} f_i^{(4)}$$

$$+ f_{i-1} = f_i - h f_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{3!} f_i''' + \frac{h^4}{4!} f_i^{(4)}$$

$$f_i'' = \frac{1}{h^2} [f_{i+1} - 2f_i + f_{i-1}] - \frac{h^2}{12} f_i^{(4)} \rightarrow \text{sub into } \textcircled{1}$$

$$I_i = 2h f_i + \frac{h}{3} f_{i+1} - \frac{2h}{3} f_i + \frac{h}{3} f_{i-1} - \frac{h^5}{36} f_i^{(4)} + \frac{h^5}{60} f_i^{(5)}$$

$$I_i = \frac{h}{3} [P_{i-1} + 4P_i + P_{i+1}] - \frac{h^5}{90} P^{(4)}(\xi) \quad \hookrightarrow \text{T.E}$$

$$I = \sum_{i \rightarrow \text{odd}} I_i = I_1 + I_3 + \dots + I_{\frac{n}{2}+1}$$

$$I = \frac{h}{3} [P_0 + 4P_1 + P_2] + \frac{h}{3} [P_2 + 4P_3 + P_4] + \dots - \frac{mh^5}{90} P^{(4)}(\xi) \quad \hookrightarrow \text{T.E}$$

(لأن كل ثلاث نقاط يمثل فترة) m -intervals

$$= \frac{h}{3} [P_0 + P_n + 4(P_1 + P_3 + \dots) + 2(P_2 + P_4 + \dots)] - \frac{mh^5}{90} P^{(4)}(\xi)$$

$$m = \frac{n}{2}, \quad h = \frac{b-a}{n}$$

$$M_4 = \max_{\xi \in [x_0, x_n]} |P^{(4)}(\xi)|$$

$$\text{T.E} \leq \frac{nh^5}{180} M_4 = \frac{(b-a)^5}{180n^4} M_4 \quad \#$$

Romberg.

$$R_{K,1} = \frac{b-a}{2^K} \left[f(a) + f(b) \right] + \frac{b-a}{2^{K-1}} \sum_{i=1}^{2^{K-1}-1} f\left(a + \frac{b-a}{2^{K-1}} i\right)$$

$$R_{K,m} = \frac{4^{m-1}}{4^{m-1}-1} R_{K,m-1} + \frac{1}{4^{m-1}-1} R_{K-1,m-1}$$

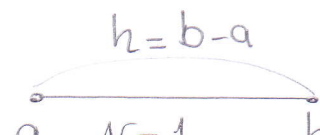
where $K=1, 2, \dots, N-1$, $N=2^{k-1}$, $h = \frac{b-a}{N} = \frac{b-a}{2^{K-1}}$

from Trapezoidal Integration

$$I = \int_a^b f(x) dx = \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{N-1} f(x_i) + f(b) \right] - \underbrace{\frac{b-a}{12} f''(\xi) h^2}_{\substack{\text{اعترافاً ثوابت} \\ \downarrow T.E}}$$

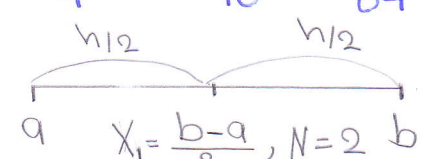
$$I = \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{N-1} f(x_i) + f(b) \right] + \underbrace{C_2 h^2 + C_4 h^4 + C_6 h^6 + \dots}_{O(h^2) \text{ خطأ بدلالة } h} \rightarrow \textcircled{1}$$

at $K=1 \rightarrow N=2^1=2$, $h=b-a \rightarrow$ sub. into $\textcircled{1}$

$$I = \frac{b-a}{2} [f(a) + f(b)] + C_2 h^2 + C_4 h^4 + \dots$$


$$I = R_{11} + C_2 h^2 + C_4 h^4 + C_6 h^6 + \dots \rightarrow \textcircled{2}$$

Replace h by $h/2$ $\Rightarrow N=2 \rightarrow K=2$ sub into $\textcircled{1}$

$$I = \frac{b-a}{\textcircled{4}} \left[f(a) + \textcircled{2} \sum_{i=1}^1 f(x_i) + f(b) \right] + C_2 \frac{h^2}{4} + C_4 \frac{h^4}{16} + C_6 \frac{h^6}{64} + \dots$$


$$I = \frac{b-a}{4} \left[f(a) + f(b) \right] + \frac{b-a}{2} \left[f\left(a + \frac{b-a}{2}\right) \right] + C_2 \frac{h^2}{4} + C_4 \frac{h^4}{16} + \dots$$

$\xrightarrow{\quad R_{21} \quad}$

$$I = R_{21} + C_2 \frac{h^2}{4} + C_4 \frac{h^4}{16} + C_6 \frac{h^6}{64} + \dots \rightarrow (3)$$

$$4 \times (3) - (2)$$

$$4I - I = 4R_{21} - R_{11} + C_2 h^4 - C_2 h^2 + C_4 \frac{h^4}{4} - C_4 h^4$$

$$3I = 4R_{21} - R_{11} - \frac{3}{4} C_4 h^4 + \dots$$

$$I = \frac{4}{3} R_{21} - \frac{1}{3} R_{11} - \frac{1}{4} C_4 h^4 + \dots$$

$\underbrace{\hspace{1.5cm}}_{R_{22}}$

$$I = R_{22} - \frac{1}{4} C_4 h^4 + \dots \rightarrow (4)$$

Replace h by $h/4$ or h by $h/2$ in the last formula (4)

$$I = \frac{4}{3} R_{31} - \frac{1}{3} R_{21} - \frac{1}{4} C_4 (h/2)^4$$

$\underbrace{\hspace{1.5cm}}_{R_{33}}$

$$I = R_{33} - \frac{1}{64} C_4 h^4 + \dots \rightarrow (5)$$

$$16 \times (5) - (4)$$

$$16I - I = 16R_{33} - R_{22} - \frac{1}{4} C_4 h^4 + \frac{1}{4} C_4 h^4 + o(h^6)$$

$$I = \frac{16}{15} R_{33} - \frac{1}{15} R_{22} + o(h^6)$$

$\underbrace{\hspace{1.5cm}}_{R_{32}} \quad \text{and so on} \quad \#$

Euler's Global Error:-

$$\text{error} = \text{exact} - \text{app.} \rightarrow e_n = y(x_n) - y_n$$

From Taylor

$$y(x_{n+1}) = y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(\xi) \quad \hookrightarrow \text{T.E}$$

$$e_{n+1} = y(x_{n+1}) - y_{n+1}$$

$$y(x_{n+1}) - y_{n+1} = y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(\xi) - y_{n+1}$$

$\hookrightarrow f(x_n, y(x_n)) \quad \hookrightarrow y_n + h f(x_n, y_n)$

$$e_{n+1} = \underbrace{y(x_n) - y_n}_{e_n} + h [f(x_n, y(x_n)) - f(x_n, y_n)] + \frac{h^2}{2!} y''(\xi)$$

$$e_{n+1} = e_n + h \left[f(x_n, y(x_n)) - f(x_n, y_n) \right] \times \frac{y(x_n) - y_n}{y(x_n) - y_n} + \frac{h^2}{2!} y''(\xi)$$

$\Delta y \rightarrow 0$ باعتبار y اتجاه Δy

$$e_{n+1} = e_n + h \frac{\partial f}{\partial y} e_n + \frac{h^2}{2!} y''(\xi)$$

$$|y''| < K, \quad |f_y| < M \quad (\text{max at the interval})$$

$$|e_{n+1}| \leq |e_n| + h |f_y| |e_n| + \frac{h^2}{2} |y''|$$

$$|e_{n+1}| \leq |e_n| [1 + hM] + \frac{h^2}{2} K \rightarrow \text{let } c = \frac{hK}{2M} \rightarrow K = \frac{2Mc}{h}$$

$$|e_{n+1}| \leq |e_n| [1 + hM] + hMc$$

$$\text{at } n=0 \quad e_0 = y(x_0) - y_0 = 0 \quad \text{initial is the exact}$$

$$\therefore |e_1| \leq hMc$$

at $n=1$

$$|e_2| \leq (1+hM)|e_1| + hMc$$

$$\leq (1+hM)hMc + hMc = c[h^2M^2 + 2hM + 1 - 1] \\ = c[(hM+1)^2 - 1]$$

$$\therefore |e_2| \leq c[(hM+1)^2 - 1]$$

$$\therefore |e_n| \leq c[(hM+1)^n - 1]$$

$$h = \frac{b-a}{n}, \quad C = \frac{hK}{2M}$$

$$K = \max_{x_0 \leq \xi \leq x_n} |y''(\xi)|, \quad M = \max_{x_0 \leq \eta \leq x_n} |f_{yy}(\eta)|$$

“ $a \rightarrow \tilde{p}$ ”